# A $q$-enumeration of generalized plane partitions

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**Abstract**

MacMahon proved a simple product formula for the generating function of the plane partitions fitting in a given rectangular box. The theorem implies the number of lozenge tilings of a semi-regular hexagon on the triangular lattice. By investigating the lozenge tilings of a hexagon with a hole on the boundary, we generalize the ordinary plane partitions to piles of unit cubes fitting in a union of several adjacent rectangular boxes. We extend MacMahon's classical theorem by proving that the generating function of the generalized plane partitions is given by a simple product formula.

**Keywords:** perfect matchings, plane partitions, lozenge tilings, dual graph, graphical condensation.

## 1 Introduction and main results

Given $k$ positive integers $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k$. A *plane partition* of shape $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is an array of non-negative integers

$$\begin{array}{ccccccc} n_{1,1} & n_{1,2} & n_{1,3} & \cdots & \cdots & \cdots & n_{1,\lambda_1} \\ n_{2,1} & n_{2,2} & n_{2,3} & \cdots & \cdots & n_{2,\lambda_2} & \\ \vdots & \vdots & \vdots & \vdots & \iddots & & \\ n_{k,1} & n_{k,2} & n_{k,3} & \cdots & n_{k,\lambda_k} & & \end{array}$$

so that $n_{i,j} \geqslant n_{i,j+1}$ and $n_{i,j} \geqslant n_{i+1,j}$ (i.e. all rows and all columns are weakly decreasing from left to right and from top to bottom, respectively). The sum of all entries of a plane partition $\pi$ is called the *volume* (or the *norm*) of the plane partition, and denoted by $|\pi|$.

The plane partitions of rectangular shape $(b, b, \ldots, b)$ ($a$ rows) with entries at most $c$ are usually identified with their 3-D interpretations — piles (or stacks) of unit cubes fitting in an

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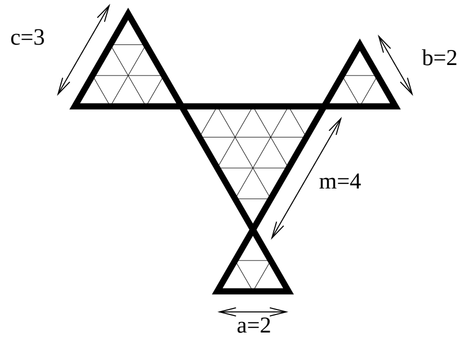



Figure 1.1: The shamrock $S_{4,2,2,3}$.

$a \times b \times c$ box. The latter are in bijection with *lozenge tilings* of a semi-regular hexagon of side-lengths $a, b, c, a, b, c$ (in clockwise order; starting from the northwest side[1]) on the triangular lattice. Here, a *lozenge* (or *unit rhombus*) is union of any two unit equilateral triangles sharing an edge; and a *lozenge tiling* of a region is a covering of the region by lozenges so that there are no gaps or overlaps.

Let $q$ be an indeterminate. The *q-integer* $[n]_q$ is defined by $[n]_q := 1 + q + q^2 + \ldots + q^{n-1}$. We also define the *q-factorial* by $[n]_q! := [1]_q[2]_q \ldots [a]_q$, and the *q-hyperfactorial* function by $\mathrm{H}_q(n) := [0]_q! \cdot [1]_q! \cdot [2]_q! \ldots [n-1]_q!$. MacMahon [Ma] proved that the volume generating function of the plane partitions fitting in an $a \times b \times c$ box is given by

$$\sum_{\pi} q^{|\pi|} = \frac{\mathrm{H}_q(a)\,\mathrm{H}_q(b)\,\mathrm{H}_q(c)\,\mathrm{H}_q(a+b+c)}{\mathrm{H}_q(a+b)\,\mathrm{H}_q(b+c)\,\mathrm{H}_q(c+a)}. \tag{1.1}$$

The $q = 1$ specialization of the Macmahon formula (1.1) is equivalent to the fact that the number of lozenge tilings of a semi-regular hexagon $Hex(a, b, c)$ of side-lengths $a, b, c, a, b, c$ is equal to

$$\frac{\mathrm{H}(a)\,\mathrm{H}(b)\,\mathrm{H}(c)\,\mathrm{H}(a+b+c)}{\mathrm{H}(a+b)\,\mathrm{H}(b+c)\,\mathrm{H}(c+a)}, \tag{1.2}$$

where $\mathrm{H}(n) := \mathrm{H}_1(n) = 0! \cdot 1! \cdot 2! \ldots (n-1)!$ is the ordinary hyperfactorial function.

Extending the MacMahon's classical theorem (for the case when $q = 1$), Ciucu, Eisenkölbl, Krattenthaler, and Zare [CEKZ] proved a simple product formula for the number of tilings of a hexagon of side-lengths $a, b+m, c, a+m, b, c+m$ with a triangular hole of size $m$ in the center (the region was called *cored hexagon* in [CK]). Recently, Ciucu and Krattenthaler [CK] generalized the latter result further by extending the central triangular hole to a *shamrock hole* consisting of four adjacent equilateral triangles, and showing that the tiling formula is still a simple product formula. Precisely, the *shamrock* $S_{m,a,b,c}$ is the union of four equilateral triangles with sides $m$, $a$, $b$, $c$ on the triangular lattice described in Figure 1.1.

Let us consider a related situation of the cored hexagon in [CEKZ] when the triangular hole appears on the boundary (instead of in the center). It has been proven that the regions of such type have the number of lozenge tilings given by a simple product formula (see e.g. Proposition 2.1 in [CLP]). As suggested by Ciucu and Krattenthaler's work in [CK], we extend the triangular hole on the boundary of the hexagon to a shamrock hole as follows.

[1]From now on, we always list the side-lengths of a hexagon on the triangular lattice in the clockwise order, starting from the northwest side.

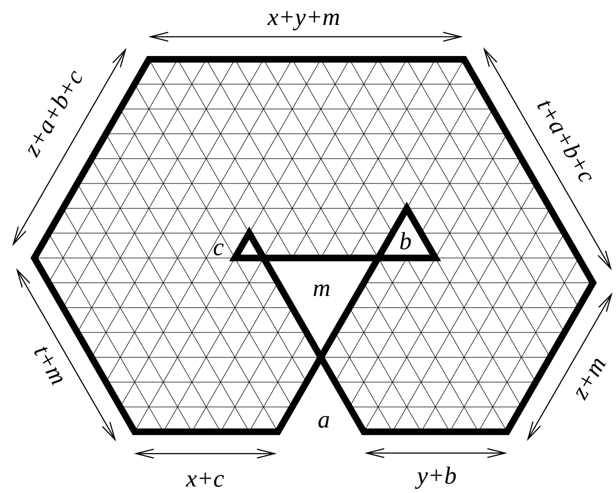



Figure 1.2: Hexagon with a shamrock removed along the boundary.

We start with a hexagon of side-lengths $z+a+b+c, x+y+m, t+a+b+c, z+m, x+y+a+b+c, t+m$. Next, we remove a shamrock $S_{m,a,b,c}$ from the base of the hexagon so that the lower-left vertex of the $a$-triangle in the shamrock is $x+c$ units to the right of the lower-left vertex of the hexagon. We denote by $Q\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix}$ the resulting region. Figure 1.2 shows the region $Q\begin{pmatrix} 4 & 3 & 2 & 3 \\ 4 & 3 & 2 & 1 \end{pmatrix}$.

Interestingly, the lozenge tilings of our new region are always enumerated by a simple product formula.

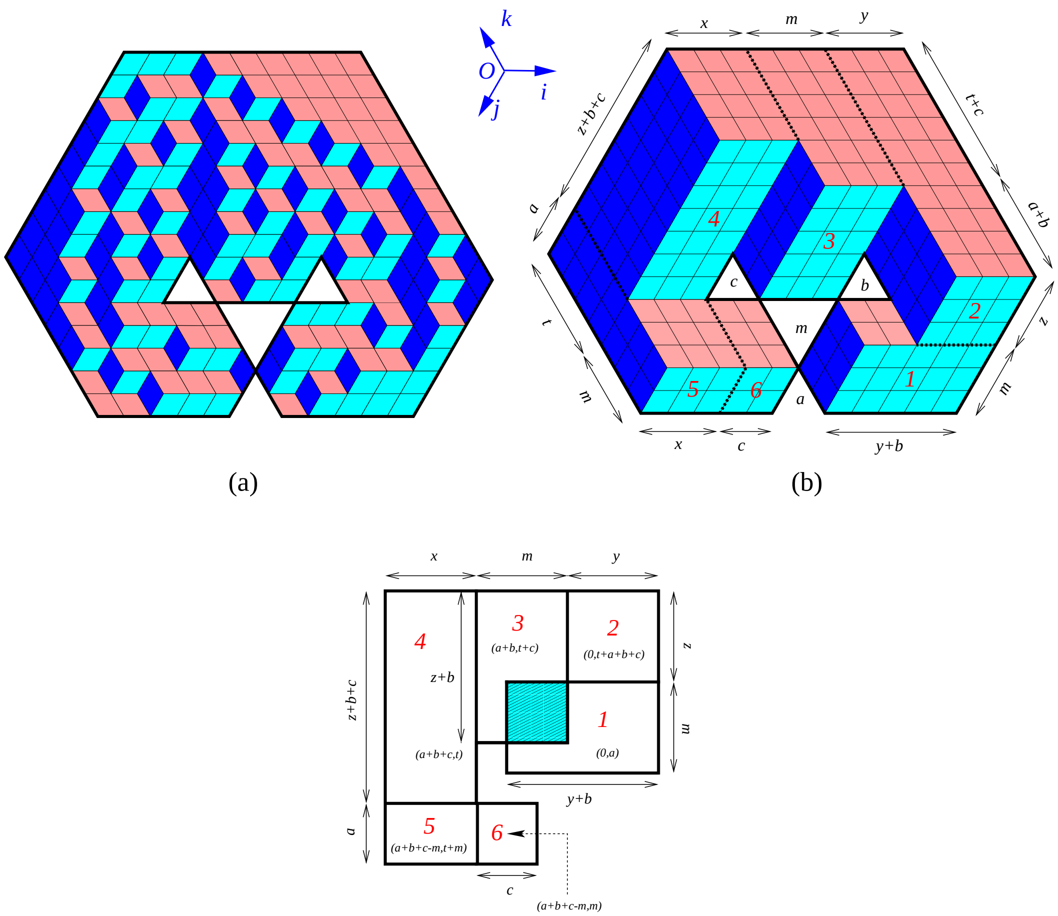
**Theorem 1.1.** *For non-negative integers $x, y, z, t, m, a, b, c$*

$$\begin{aligned}
\mathrm{M}\left(Q\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix}\right) &= \frac{\mathrm{H}(m+a+b+c+x+y+z+t)}{\mathrm{H}(m+a+b+c+x+y+t)\,\mathrm{H}(m+a+b+c+x+y+z)} \\
&\times \frac{\mathrm{H}(m+a+b+c+x+t)\,\mathrm{H}(m+a+b+c+x+y)\,\mathrm{H}(m+a+b+c+y+z)}{\mathrm{H}(m+a+b+c+z+t)\,\mathrm{H}(m+a+b+c+x)\,\mathrm{H}(m+a+b+c+y)} \\
&\times \frac{\mathrm{H}(x)\,\mathrm{H}(y)\,\mathrm{H}(z)\,\mathrm{H}(t)}{\mathrm{H}(x+t)\,\mathrm{H}(y+z)} \frac{\mathrm{H}(m)^3\,\mathrm{H}(a)^2\,\mathrm{H}(b)\,\mathrm{H}(c)\,\mathrm{H}(m+a+b+c)}{\mathrm{H}(m+a)^2\,\mathrm{H}(m+b)\,\mathrm{H}(m+c)} \\
&\times \frac{\mathrm{H}(m+b+c+z+t)\,\mathrm{H}(m+a+c+x)\,\mathrm{H}(m+a+b+y)}{\mathrm{H}(m+b+y+z)\,\mathrm{H}(m+c+x+t)} \\
&\times \frac{\mathrm{H}(c+x+t)\,\mathrm{H}(b+y+z)}{\mathrm{H}(a+c+x)\,\mathrm{H}(a+b+y)\,\mathrm{H}(b+c+z+t)},
\end{aligned} \tag{1.3}$$

*where we use the notation $\mathrm{M}(R)$ for the number of lozenge tilings of a region $R$*

Next, we consider a $q$-analog of Theorem 1.1.

Similar to the bijection between lozenge tilings of a semi-regular hexagon $Hex(a, b, c)$ and plane partitions fitting in an $a \times b \times c$ box, one can view a lozenge tiling of $Q\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix}$ as




(a) (b)

(c)

Figure 1.3: (a) Viewing a lozenge tiling of the region as a pile of unit cubes in a special box. (b) The tiling corresponding to the empty pile. (c) The projection of the compound box $\mathcal{B}$ on the **Oij** plane.

a pile of unit cubes that fits in a *compound box* $\mathcal{B} := \mathcal{B}\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix}$, which is the union of 6 adjacent rectangular boxes (see Figure 1.3(a) for $x = y = z = 3, t = 4, m = 3, a = b = c = 2$). Figure 1.3(b) gives a 3-D picture of the compound box $\mathcal{B}$ by showing the empty pile; the bases of the rectangular boxes in $\mathcal{B}$ are labeled by $1, 2, \ldots, 6$.

Projecting the compound box $\mathcal{B}$ on the **Oij** plane, we get a projective diagram as in Figure 1.3(c). In this diagram, each rectangular box in $\mathcal{B}$ is represented by a rectangle with a pair of integers $(a, b)$, where $a$ is the level of the base and $b$ is the height of the box. We always assume that the base of the box 1 is on level 0. We also note that the rectangles corresponding to the boxes 1 and 3 are overlapped (the intersection is indicated by the shaded area in Figure 1.3(c)). However, these two boxes are *not* overlapped since the base of the box 3 is above the top of the box 1 (as $a + b \geqslant a$).

We call the piles of unit cubes fitting in the compound box $\mathcal{B}$ *generalized plane partitions*, since they have a similar monotonicity as (the 3D-interpretation of) the ordinary plane partitions:

the tops of their columns (of unit cubes) are weakly decreasing along $\overrightarrow{\mathbf{Oi}}$ and $\overrightarrow{\mathbf{Oj}}$.

Similar to MacMahon's theorem (1.1), we have a closed form product formula for the volume generating function of the generalized plane partitions.

**Theorem 1.2.** *Let $m, a, b, c, x, y, z, t$ be non-negative integers. Then*

$$\begin{aligned}\sum_{\pi} q^{|\pi|} &= \frac{\mathrm{H}_q(m+a+b+c+x+y+z+t)}{\mathrm{H}_q(m+a+b+c+x+y+t)\,\mathrm{H}_q(m+a+b+c+x+y+z)}\\
&\times \frac{\mathrm{H}_q(m+a+b+c+x+t)\,\mathrm{H}_q(m+a+b+c+x+y)\,\mathrm{H}_q(m+a+b+c+y+z)}{\mathrm{H}_q(m+a+b+c+z+t)\,\mathrm{H}_q(m+a+b+c+x)\,\mathrm{H}_q(m+a+b+c+y)}\\
&\times \frac{\mathrm{H}_q(x)\,\mathrm{H}_q(y)\,\mathrm{H}_q(z)\,\mathrm{H}_q(t)}{\mathrm{H}_q(x+t)\,\mathrm{H}_q(y+z)}\frac{\mathrm{H}_q(m)^3\,\mathrm{H}_q(a)^2\,\mathrm{H}_q(b)\,\mathrm{H}_q(c)\,\mathrm{H}_q(m+a+b+c)}{\mathrm{H}_q(m+a)^2\,\mathrm{H}_q(m+b)\,\mathrm{H}_q(m+c)}\\
&\times \frac{\mathrm{H}_q(m+b+c+z+t)\,\mathrm{H}_q(m+a+c+x)\,\mathrm{H}_q(m+a+b+y)}{\mathrm{H}_q(m+b+y+z)\,\mathrm{H}_q(m+c+x+t)}\\
&\times \frac{\mathrm{H}_q(c+x+t)\,\mathrm{H}_q(b+y+z)}{\mathrm{H}_q(a+c+x)\,\mathrm{H}_q(a+b+y)\,\mathrm{H}_q(b+c+z+t)},\end{aligned} \tag{1.4}$$

*where the sum on the left-hand side is taken over all generalized plane partitions $\pi$ fitting in the compound box $\mathcal{B}\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}$, and where $|\pi|$ is the volume of $\pi$ (i.e. the number of unit cubes in $\pi$).*

One readily sees that Theorem 1.1 is a consequence of Theorem 1.2. Moreover, by letting $m = a = b = c = 0$ (i.e. the shamrock hole is empty), we get the MacMahon's classical theorem (1.1) from Theorem 1.2.

The goal of the present paper is to prove Theorem 1.2 by using the graphical condensation method first introduced by Eric H. Kuo in [Ku04]. This condensation can be viewed as a combinatorial interpretation of Dodgson condensation (also called as the Jacobi-Desnannot identity, see e.g. [Mu], pp.136–149). We refer the reader to e.g. [YYZ], [YZ], [Ku06], [Sp], [Ci], [Fu] for various aspects and generalizations of the method; and e.g. [CK], [CL], [CF14], [CF15], [KW], [La15a], [La15b], [La15c], [LMNT], [Zh] for recent applications of Kuo condensation.

The rest of our paper is organized as follows. For ease of reference, we quote the several preliminary results in Section 2, including the particular version of Kuo condensation employed in our proofs. In order to apply Kuo condensation to our $Q$-type regions, we consider several simple weight assignments on the lozenges of the regions in Section 3. Section 4 generalizes a related work of Ciucu and Krattenthaler in [CK, Theorem 3.1] by $q$-enumerating lozenge tilings of a *magnet bar region* (the $b = c = 0$ specialization of a $Q$-type region). Finally, we prove Theorem 1.2 in Section 5.

# 2 Preliminaries

Let $G$ be a finite simple graph without loops. A *perfect matching* of $G$ is a collection of disjoint edges covering all vertices of $G$. Let $R$ be a *region*[2]. The *(planar) dual graph* of $R$ is the graph whose vertices are unit triangles in $R$ and whose edges connect precisely two unit triangles

[2] From now on, we use the word *region* to mean a finite connected region on the triangular lattice.

sharing an edge. One can identify the lozenge tilings of $R$ with the perfect matchings of its dual graph.

For a weighted graph $G$, we define the *matching generating function* $\mathrm{M}(G)$ of $G$ to be the sum of the weights of all perfect matchings in $G$, where the *weight* of a perfect matching is the product of weights of its edges. If the lozenges of a region $R$ are weighted, we define similarly the *tiling generating function* $\mathrm{M}(R)$ of $R$. We notice that each edge of the dual graph $G$ of the region $R$ carries the same weight as its corresponding lozenge in $R$.

The following Kuo's condensation theorem is the key of our proofs.

**Theorem 2.1** (Theorem 5.1 in [Ku04])**.** *Let $G = (V_1, V_2, E)$ be a (weighted) bipartite planar graph in which $|V_1| = |V_2|$. Assume that $u, v, w, s$ are four vertices appearing in a cyclic order on a face of $G$ so that $u, w \in V_1$ and $v, s \in V_2$. Then*

$$\mathrm{M}(G)\,\mathrm{M}(G-\{u,v,w,s\}) = \mathrm{M}(G-\{u,v\})\,\mathrm{M}(G-\{w,s\}) + \mathrm{M}(G-\{u,s\})\,\mathrm{M}(G-\{v,w\}). \tag{2.1}$$

A *forced lozenge* of a region $R$ is a lozenge contained in any tilings of $R$. Assume that we removed several forced lozenges $\ell_1, \ell_2, \dots, \ell_n$ from the region $R$, and denote by $R'$ the resulting region. Then one clearly has

$$\mathrm{M}(R) = \mathrm{M}(R') \prod_{i=1}^{n} wt(\ell_i), \tag{2.2}$$

where $wt(\ell_i)$ denotes the weight of the lozenge $\ell_i$.

If a region $R$ admits a lozenge tiling, then the numbers of up-pointing unit triangles and down-pointing unit triangles in $R$ are equal. Moreover, if a region satisfies the latter balancing condition, we say that the region is *balanced*. The following lemma can be considered as a generalization of the identity (2.2).

**Lemma 2.2** (Region-splitting Lemma)**.** *Let $R$ be a balanced region. Assume that a subregion $S$ of $R$ satisfies following two conditions:*

*(i)* (Separating Condition) *There are only one type of unit triangles (up-pointing or down-pointing) running along each side of the border between $S$ and $R - S$.*

*(ii)* (Balancing Condition) *$S$ is balanced.*

*Then*

$$\mathrm{M}(R) = \mathrm{M}(S)\ \mathrm{M}(R - S). \tag{2.3}$$

*Proof.* Let $G$ be the dual graph of the region $R$. Then the dual graph $K$ of the subregion $S$ is an induced subgraph of $G$. It is easy to see that $K$ satisfies the conditions in Lemma 3.6(a) in [La14], and the lemma follows. □

# 3 $q$-weight assignments

Lozenges in a region $R$ come with three different orientations: left, right, and vertical lozenges (see Figure 3.1). Next, we consider three simple $q$-weight assignments of lozenges in our region $Q := Q\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix}$ as follows.

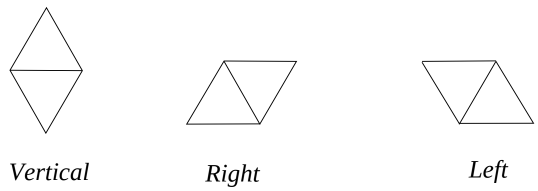



Figure 3.1: Three orientations of lozenges.

View a lozenge tiling $T$ of the region $Q$ as a generalized plane partition (i.e. a pile of unit cubes); each right lozenge is viewed as the top of a column of unit cubes. We assign to each right lozenge a weight $q^x$, where $x$ is the number of unit cubes in the corresponding column. All left and vertical lozenge are weighted by 1. We call this assignment the *natural q-weight assignment* of $Q$, and use the notation $wt_0$ for the assignment (see Figure 3.2(a) for $x = y = z = 3$, $t = 4$, $m = 3$, $a = b = c = 2$).

Besides the natural $q$-weight assignment $wt_0$, we consider the following two simple weight assignments:

(1) *Assignment 1.* The weights of left and vertical lozenges are all 1. The weight of a right lozenge is $q^l$, where $l$ is the distance between the left side of the lozenge and the southeast side of the region $Q$. We use notation $wt_1$ for this weight assignment (see Figure 3.2(b)).

(2) *Assignment 2.* All left and vertical lozenges are still weighted by 1. However, a right lozenge is now weighted by $q^n$, where $n$ is the distance between the top of the lozenge and the base of the region $Q$. This assignment is denoted by $wt_2$ (see Figure 3.2(c)).

Let $T$ be a tiling of $Q$. We denote by $wt_0(T)$, $wt_1(T)$ and $wt_2(T)$ the weights of the tiling $T$ with respect to the weight assignments $wt_0$, $wt_1$ and $wt_2$. We also denote by $\mathrm{M}_0(Q)$, $\mathrm{M}_1(Q)$ and $\mathrm{M}_2(Q)$ the tiling generating functions of $Q$ corresponding to the weight assignments $wt_0$, $wt_1$ and $wt_2$. It is easy to see that $\mathrm{M}_0(Q)$ is exactly the volume generation function of the generalized plane partitions corresponding to the lozenge tilings of $Q$.

In the next proposition, we will show that the three weight assignments are the same up to some multiplicative factors.

We define two functions

$$\mathbf{f}\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix} := m\binom{y+b+1}{2} + z\binom{y+1}{2} + m(z+b)(y+a+b) + (z+b)\binom{m+1}{2}$$
$$+ x(z+b+c)(y+m+a+b+c) + (z+b+c)\binom{x+1}{2} + a(x+c)(y+a+b) + a\binom{x+c+1}{2} \tag{3.1}$$

and

$$\mathbf{g}\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix} := (y+b)\binom{m+1}{2} + myz + y\binom{z+1}{2} + m(z+b)(m+a)$$
$$+ m\binom{z+b+1}{2} + x(m+a)(z+b+c) + x\binom{z+b+c+1}{2} + (x+c)\binom{a+1}{2}. \tag{3.2}$$

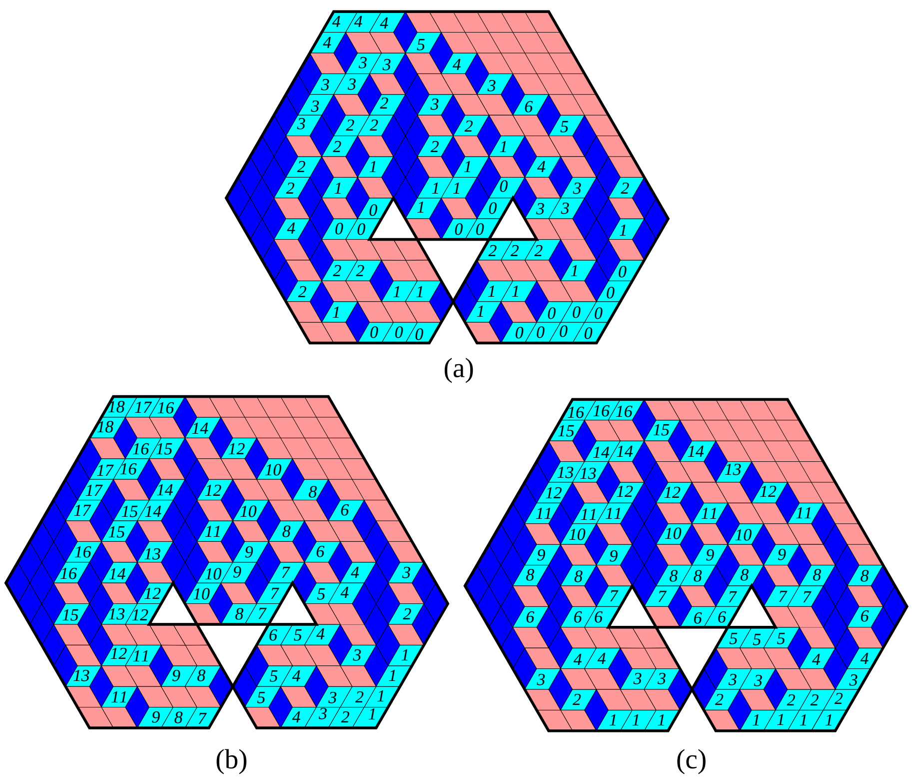


Figure 3.2: Three $q$-weight assignments on a sample tiling of region $Q$: (a) $wt_0$, (b) $wt_1$, (c) $wt_2$. The right lozenges with label $x$ are weighted by $q^x$.

**Proposition 3.1.** *For any non-negative integers $m, a, b, c, x, y, z, t$*

$$\mathrm{M}_1\left(Q\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}\right) = q^{\boldsymbol{f}\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}} \sum_{\pi} q^{|\pi|} \tag{3.3}$$

*and*

$$\mathrm{M}_2\left(Q\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}\right) = q^{\boldsymbol{g}\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}} \sum_{\pi} q^{|\pi|}, \tag{3.4}$$

*where the sums on the right-hand sides are taken over all generalized plane partitions $\pi$ fitting in the compound box $\mathcal{B}\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}$.*

*Proof.* We use the following shorthand notations in this proof: $\mathbf{f} := \mathbf{f}\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}$, $\mathbf{g} := \mathbf{g}\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}$, $\mathcal{B} := \mathcal{B}\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}$, and $Q := Q\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}$.

Let $T$ be any lozenge tiling of the region $Q$, and $\pi$ the generalized plane partition corresponding to $T$. We only need to show that

$$\frac{wt_1(T)}{q^{|\pi|}} = q^{\mathbf{f}} \quad \text{and} \quad \frac{wt_2(T)}{q^{|\pi|}} = q^{\mathbf{g}}. \tag{3.5}$$

Assume that the box $i$ of the compound box $\mathcal{B}$ has size $a_i \times b_i \times c_i$ (for $1 \leqslant i \leqslant 6$). The base of the box $i$ is pictured as a parallelogram $P_i$ in Figure 1.3(b). We assume in addition that the left side of $P_i$ is $x_i$ units to the left of the southeast side of the region $Q$, and the bottom of $P_i$ is $y_i$ units above the bottom of region $Q$.

Divide the generalized plane partition $\pi$ into 6 disjoint sub-partitions $\pi_i$ ($1 \leqslant i \leqslant 6$) fitting in the box $i$. Each partial-partition $\pi_i$ in turn gives a lozenge tiling $T_i$ of the semi-regular hexagon $Hex(a_i, b_i, c_i)$. Figure 3.3(a) shows the partial-partition $\pi_3$ of the generalized plane partition $\pi$ in Figure 1.3(a), as well as, the relative positions of the parallelogram $P_3$ to the bottom and the southeast side of the region $Q$ (for $x = 3, y = 3, z = 3, t = 4, m = 3, a = 2, b = 2, c = 2$).

Apply the weight assignment $wt_1$ to the whole tiling $T$ of the region $Q$. This yields a local weight assignment $wt_1^{(i)}$ for the tiling $T_i$ of hexagon $Hex(a_i, b_i, c_i)$. Precisely, each right lozenge in $T_i$ is now weighted by $q^{x_i+l}$, where $l$ is the distance between the left side of the lozenge and the southeast side of the hexagon $Hex(a_i, b_i, c_i)$. Encode the tiling $T_i$ as a family of $b_i$ disjoint lozenge-paths connecting the top and the bottom of the hexagon (see the dotted paths in Figure 3.3(b)). Dividing the weight of each right lozenge in the lozenge-path $j$ (from right to left) by $q^{x_i+j}$, we get the weight assignment $wt_0$ for $T_i$. Since the end points of the above lozenge-paths are fixed, each lozenge-path has exactly $a_i$ right lozenges. Thus, we have

$$\frac{wt_1^{(i)}(T_i)}{wt_0(T_i)} = \frac{wt_1^{(i)}(T_i)}{q^{|\pi_i|}} = q^{a_ib_ix_i + a_ib_i(b_i+1)/2}.$$

Multiplying all above equations for $i = 1, 2, \dots, 6$, we get

$$\frac{wt_1(T)}{q^{|\pi|}} = q^{\sum_{i=1}^{6} a_ib_ix_i + a_ib_i(b_i+1)/2}. \tag{3.6}$$

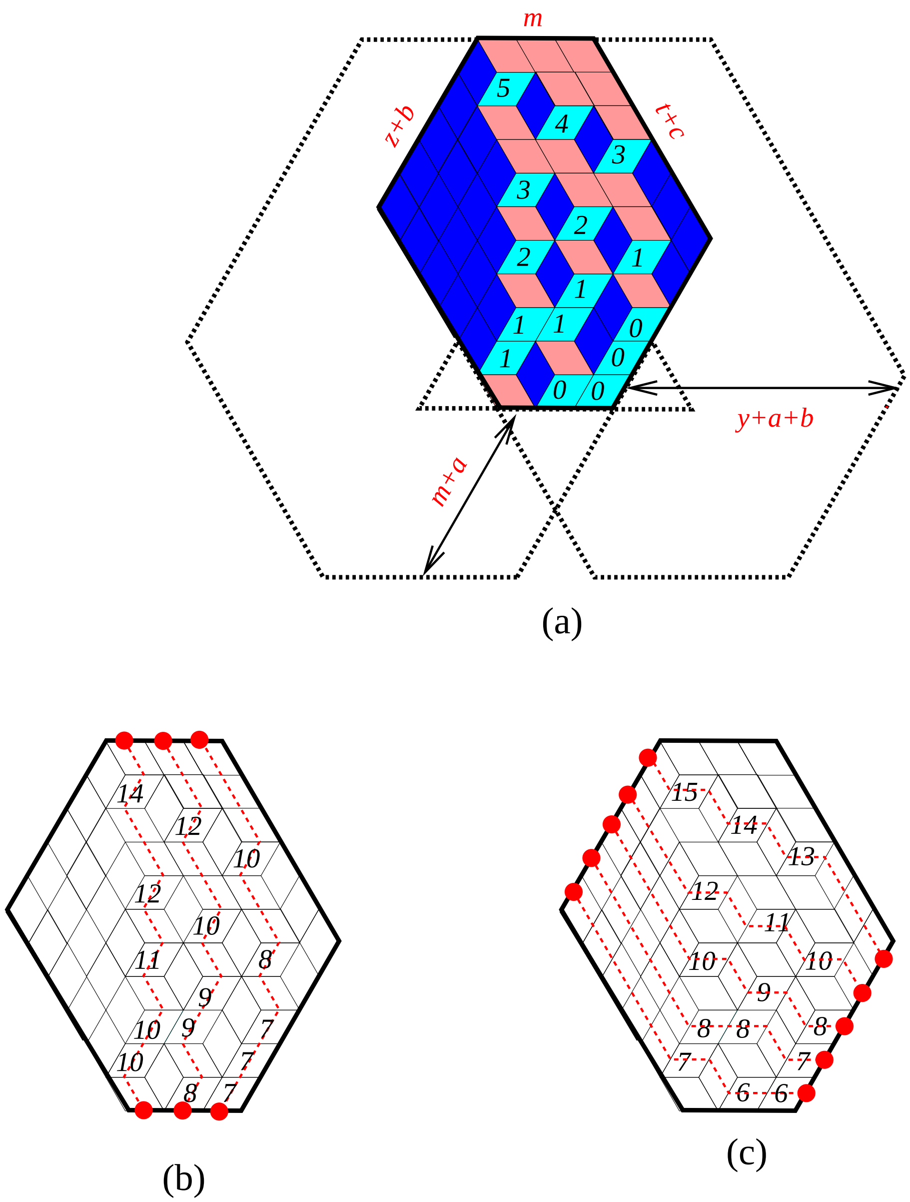



Figure 3.3: The partial-partition corresponding to the room 3.

Next, we assume that the whole tiling $T$ of $Q$ is weighted by $wt_2$. We now encode the tiling $T_i$ of $Hex(a_i, b_i, c_i)$ as an $a_i$-tuple of disjoint lozenge-paths connecting the northwest side and the southeast side of the hexagon (see Figure 3.3(c)). Dividing each right lozenge in the lozenge-path $j$ (from bottom to top) by $q^{y_i+j}$, we get back again the weight assignment $wt_0$. We note that each lozenge-path has now $b_i$ right lozenges. Similar to the case of $wt_1$, we have

$$\frac{wt_2(T)}{q^{|\pi|}} = q^{\sum_{i=1}^{6} a_i b_i y_i + b_i a_i (a_i+1)/2}. \tag{3.7}$$

Obtaining the formulas of $a_i, b_i, x_i, y_i$ in terms of $m, a, b, c, x, y, z, t$ from Figures 1.3(b) and (c), we get $\mathbf{f} = \sum_{i=1}^{6} a_i b_i x_i + a_i b_i (b_i + 1)/2$ and $\mathbf{g} = \sum_{i=1}^{6} a_i b_i y_i + b_i a_i (a_i + 1)/2$. This finishes our proof. $\square$

We note that the powers $q^{\mathbf{f}}$ and $q^{\mathbf{g}}$ in the above proposition are exactly the weights $wt_1(T_0)$ and $wt_2(T_0)$ of the tiling $T_0$ corresponding to the empty pile of unit cubes in Figure 1.3(b).

View the hexagon $Hex(a, b, c)$ as a special case of the region $Q := Q\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix}$ with an empty shamrock hole. We have the following consequence of Proposition 3.1 and MacMahon's $q$-formula (1.1).

**Corollary 3.2.** *For non-negative integers $a, b, c$*

$$\mathrm{M}_1\left(Hex(a,b,c)\right) = q^{ab(b+1)/2} \frac{\mathrm{H}_q(a)\,\mathrm{H}_q(b)\,\mathrm{H}_q(c)\,\mathrm{H}_q(a+b+c)}{\mathrm{H}_q(a+b)\,\mathrm{H}_q(b+c)\,\mathrm{H}_q(c+a)} \tag{3.8}$$

*and*

$$\mathrm{M}_2\left(Hex(a,b,c)\right) = q^{ba(a+1)/2} \frac{\mathrm{H}_q(a)\,\mathrm{H}_q(b)\,\mathrm{H}_q(c)\,\mathrm{H}_q(a+b+c)}{\mathrm{H}_q(a+b)\,\mathrm{H}_q(b+c)\,\mathrm{H}_q(c+a)}. \tag{3.9}$$

A *column-strict plane partition* is a plane partition having columns strictly decreasing. A *semihexagon* $SH_{a,b}$ is the upper half of a lozenge hexagon $Hex(a, b, a)$. We are interested in the lozenge tilings of the semihexagon $SH_{a,b}$, where $a$ up-pointing unit triangles at the positions $s_1, s_2, \dots, s_a$ have been removed from the base. Denote by $SH_{a,b}(s_1, s_2, \dots, s_a)$ the resulting *semihexagon with dents* (see Figure 3.4(a) for the region $SH_{7,5}(1, 2, 6, 7, 10, 11, 12)$). Assume that the lozenges in the semihexagon are weighted by $wt_2$, and we still use the notation $\mathrm{M}_2$ for the corresponding tiling generating function of the semihexagon with dents. There is a well-known (weight preserving) bijection between the lozenges of $SH_{a,b}(s_1, s_2, \dots, s_a)$ and the column-strict plane partitions of shape $(s_a - a, s_{a-1} - a + 1, \dots, s_1 - 1)$ with positive entries at most $a$, i.e. $wt_2(T) = q^{|\pi_T|}$, where $\pi_T$ is the plane partition corresponding to the tiling $T$ (see e.g. [CLP] and [CS]).

We have the following $q$-enumeration of the lozenge tilings of a hexagon with a triangular hole on the base $K_a(x, y, z, t)$ (defined as the region restricted by the bold contour in Figure 3.4(b)).

**Lemma 3.3.** *For non-negative $a, x, y, z, t$*

$$\mathrm{M}_2\left(K_a(x,y,z,t)\right) = q^{y\binom{z+1}{2} + x\binom{a+z+1}{2}} \frac{\mathrm{H}_q(a)\,\mathrm{H}_q(x)\,\mathrm{H}_q(y)\,\mathrm{H}_q(z)\,\mathrm{H}_q(t)}{\mathrm{H}_q(x+t)\,\mathrm{H}_q(a+x)\,\mathrm{H}_q(a+y)\,\mathrm{H}_q(y+z)}$$
$$\times \frac{\mathrm{H}_q(a+x+t)\,\mathrm{H}_q(a+x+y)\,\mathrm{H}_q(a+y+z)\,\mathrm{H}_q(a+x+y+z+t)}{\mathrm{H}_q(a+x+y+t)\,\mathrm{H}_q(a+x+y+z)\,\mathrm{H}_q(a+t+z)}. \tag{3.10}$$

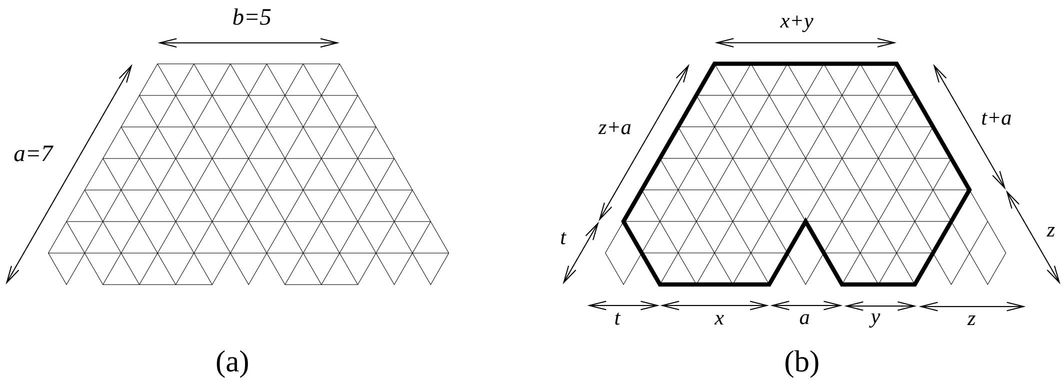



Figure 3.4: (a) The semihexagon with dents $SH_{7,5}(1,2,6,7,10,11,12)$. (b) Obtaining the region $K_a(x,y,z,t)$ (restricted by the bold contour) from a semihexagon with dents.

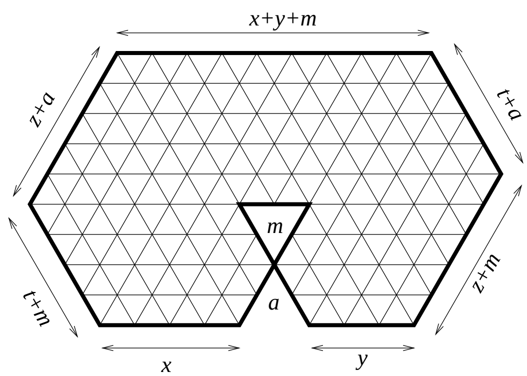



Figure 4.1: The magnet bar $B_{2,2}(4,3,3,2)$.

*Proof.* By the above bijection between lozenge tilings of the semihexagon and the column-strict plane partitions, we have

$$\mathrm{M}_2\left(SH_{a,b}(s_1,s_2,\dots,s_a)\right)=\sum_{\mu}q^{|\mu|}=q^{\sum_{i=1}^{a}(s_i-i)}\prod_{1\leqslant i<j\leqslant a}\frac{q^{s_j}-q^{s_i}}{q^j-q^i},\tag{3.11}$$

where the sum after the first equality sign is taken over all column-strict plane partitions $\mu$ of shape $(s_a-a,s_{a-1}-a+1,\dots,s_1-1)$ with positive entries at most $a$, for the second equality see e.g. [St, pp. 374–375].

The region $K_a(x,y,z,t)$ is obtained by removing forced vertical lozenges from the semihexagon $SH_{a+z+t,x+y}$ with dents at the positions $\{1,2,..,t\}\cup\{t+x+1,t+x+2,\dots,t+x+a\}\cup\{t+x+a+y+1,t+x+a+y+2,\dots,t+x+a+y+z\}$. Thus, our lemma follows from (3.11). □

# 4 Two $q$-enumerations of magnet bar regions

When $b=c=0$, our region $Q\begin{pmatrix}x & y & z & t\\ m & a & b & c\end{pmatrix}$ becomes a *magnet bar region* $B_{m,a}(x,y,z,t)$ first introduced in [CK]. Figure 4.1 shows the magnet bar region $B_{2,2}(4,3,3,2)$. In this section, we

($q$-)enumerate lozenge tilings of the magnet bar region $B_{m,a}(x,y,z,t)$. Our proof of Theorem 1.2 will use these $q$-enumerations.

**Proposition 4.1.** *For non-negative integers $m,a,x,y,z,t$*

$$\begin{aligned}\mathrm{M}_2\left(B_{m,a}(x,y,z,t)\right) &= q^{y\binom{m+1}{2}+(m+x+y)\binom{z+1}{2}+myz+(m+a)(x+m)z+x\binom{a+1}{2}} \\ &\times \frac{\mathrm{H}_q(m+a+x+y+z+t)}{\mathrm{H}_q(m+a+x+y+t)\,\mathrm{H}_q(m+a+x+y+z)} \\ &\times \frac{\mathrm{H}_q(m+a+x+t)\,\mathrm{H}_q(m+a+x+y)\,\mathrm{H}_q(m+a+y+z)}{\mathrm{H}_q(m+a+z+t)\,\mathrm{H}_q(m+a+x)\,\mathrm{H}_q(m+a+y)} \\ &\times \frac{\mathrm{H}_q(x)\,\mathrm{H}_q(y)\,\mathrm{H}_q(z)\,\mathrm{H}_q(t)\,\mathrm{H}_q(m)\,\mathrm{H}_q(a)^2}{\mathrm{H}_q(a+x)\,\mathrm{H}_q(a+y)\,\mathrm{H}_q(z+t)\,\mathrm{H}_q(m+a)} \\ &\times \frac{\mathrm{H}_q(m+z+t)\,\mathrm{H}_q(m+a+x)\,\mathrm{H}_q(m+a+y)}{\mathrm{H}_q(m+y+z)\,\mathrm{H}_q(m+x+t)}. \qquad (4.1)\end{aligned}$$

*Proof.* We prove (4.1) by induction on $y+z+t$. Our base cases are the situations when $m=0$, $a=0$, $y=0$, $z=0$ or $t=0$.

Throughout this proof, we assume that our magnet bar region is weighted by $wt_2$.

If $m=0$, then our magnet bar region becomes the region $K_a(x,y,z,t)$ in Lemma 3.3, and (4.1) follows.

If $a=0$, by removing forced lozenges along the base of the region $B_{m,0}(x,y,z,t)$, we get the weighted hexagon $Hex(z,x+y+m,t)$ in which a right lozenge is weighted by $q^{m+l}$, where $l$ is the distance from the top of the lozenge to the bottom of the hexagon (see Figure 4.2(e)). By dividing the weight of each right lozenge of the hexagon by $q^m$, we get back the weight assignment $wt_2$. Since the product of weights of the forced lozenges in Figure 4.2(e) is equal to $q^{y\binom{m+1}{2}}$, we get from (2.2)

$$\mathrm{M}_2\left(B_{m,0}(x,y,z,t)\right) = q^{y\binom{m+1}{2}}q^{mz(x+y+m)}\,\mathrm{M}_2\left(Hex(z,x+y+m,t)\right), \qquad (4.2)$$

where the factor $q^{mz(x+y+m)}$ comes from the weight division. Then (4.1) follows from Corollary 3.2.

If $y=0$, after removing forced vertical lozenges (which have the weight 1), we get a new weighted region $R$ (the region restricted by the bold contour in Figure 4.2(b)). By rotating $R$ $60^\circ$ clockwise and reflecting it about a vertical line, we get the region $K_m(z,a,x,t)$ weighted by $wt_1$. Thus, we have

$$\mathrm{M}_2\left(B_{m,a}(x,0,z,t)\right) = \mathrm{M}_1\left(K_m(z,a,x,t)\right), \qquad (4.3)$$

and (4.1) follows from Proposition 3.1 and Lemma 3.3 (one can view any $K$-type region as the $m=b=c=0$ specialization of a $Q$-type region).

If $z=0$, by applying Region-splitting Lemma 2.2 as in Figure 4.2(c), we get

$$\mathrm{M}_2\left(B_{m,a}(x,y,0,t)\right) = \mathrm{M}_2\left(Hex(m,y,a)\right)M_2\Big(B_{m,a}(x,y,0,t)-Hex(m,y,a)\Big). \qquad (4.4)$$

Next, we remove the forced left lozenges from the region $B_{m,a}(x,y,0,t)-Hex(m,y,a)$ and obtain the hexagon $Hex(a,x,t+m)$ weighted by $wt_2$. Then we get

$$\mathrm{M}_2\left(B_{m,a}(x,y,0,t)\right) = \mathrm{M}_2\left(Hex(m,y,a)\right)M_2\big(Hex(a,x,t+m)\big), \qquad (4.5)$$

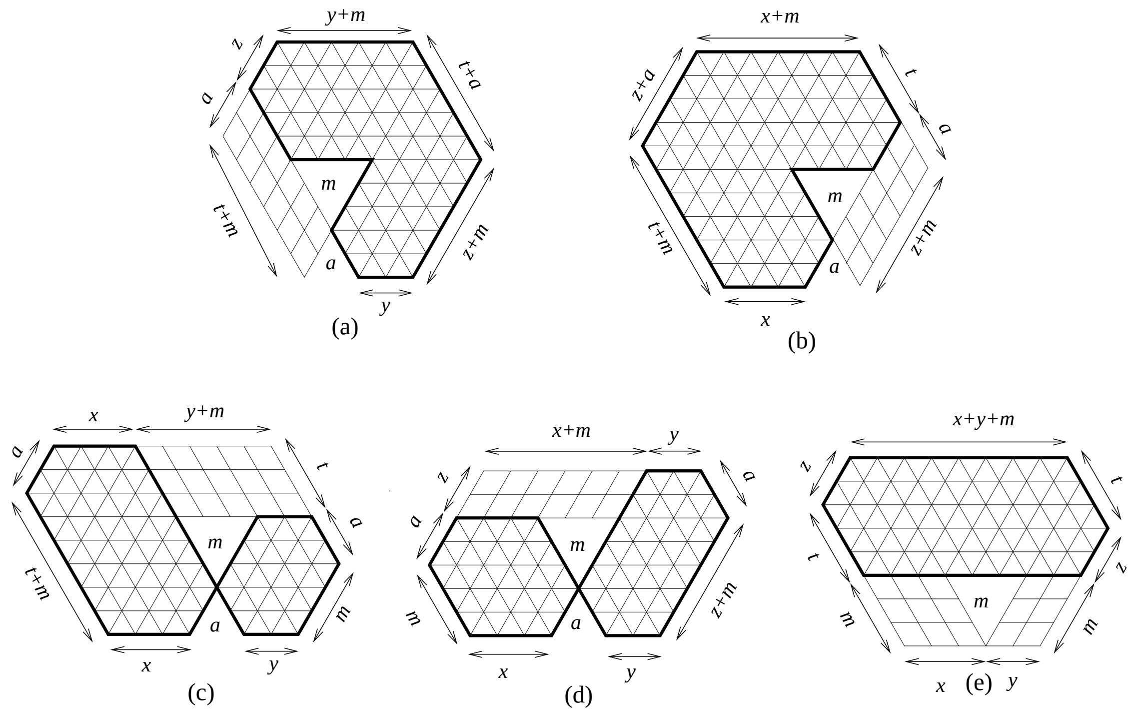



Figure 4.2: The base cases in the proofs of Propositions 4.1 and 4.2: (a) $x = 0$, (b) $y = 0$, (c) $z = 0$, (d) $t = 0$, and (e) $a = 0$.

and (4.1) follows from Corollary 3.2.

If $t = 0$, similar to the case when $z = 0$, the Region-splitting Lemma 2.2 implies

$$\mathrm{M}_2\left(B_{m,a}(x, y, z, 0)\right) = \mathrm{M}_2\left(Hex(z + m, y, a)\right) \mathrm{M}_2\left(B_{m,a}(x, y, z, 0) - Hex(z + m, y, a)\right) \quad (4.6)$$

(see Figure 4.2(d)). We also get the hexagon $Hex(a, x, m)$ (weighted by $wt_2$) after removing forced lozenges from the region $B_{m,a}(x, y, z, 0) - Hex(z + m, y, a)$. However, our forced lozenges are now right lozenges, which have weight product equal to $q^{(m+a)(x+m)z+(x+m)\binom{z+1}{2}}$. Thus, we get

$$\mathrm{M}_2\left(B_{m,a}(x, y, z, 0) - Hex(z + m, y, a)\right) = q^{(m+a)(x+m)z+(x+m)\binom{z+1}{2}} \mathrm{M}_2\left(Hex(a, x, m)\right),$$

so

$$\mathrm{M}_2\left(B_{m,a}(x, y, z, 0)\right) = q^{(m+a)(x+m)z+(x+m)\binom{z+1}{2}} \mathrm{M}_2\left(Hex(z + m, y, a)\right) \mathrm{M}_2\left(Hex(a, x, m)\right). \quad (4.7)$$

Again, (4.1) is implied by Corollary 3.2.

For the induction step, we assume that $m, a, y, z, t \geqslant 1$ and that (4.1) holds for any magnet bar regions, which have the sum of the $y$-, $z$- and $t$-parameters strictly less than $y + z + t$.

We apply Kuo Theorem 2.1 to the dual graph $G$ of the magnet bar region $B_{m,a}(x, y, z, t)$ (weighted by $wt_2$). We pick the four vertices $u, v, w, s$ as in Figure 4.3(b). In particular, the four shaded unit triangles correspond to the four vertices: the shaded unit triangle corresponding

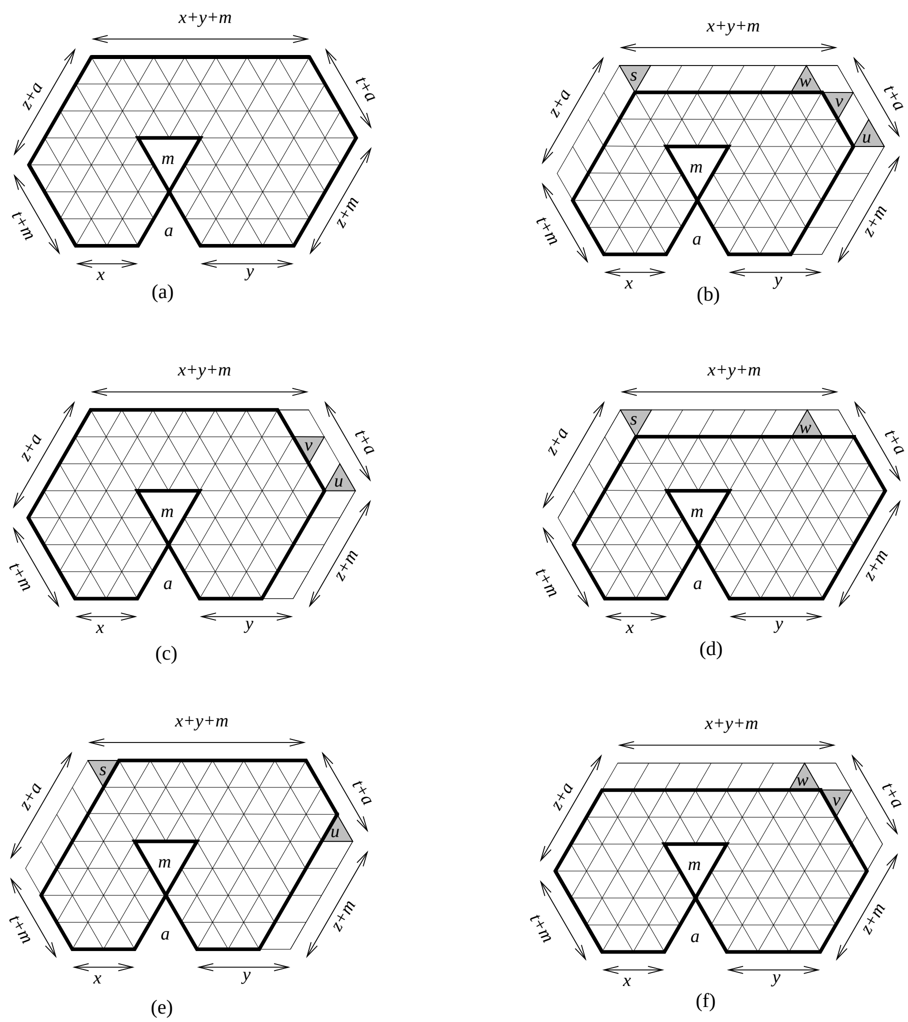



Figure 4.3: Obtaining the recurrence for the tiling generating functions of magnet bar regions.

to $u$ is the lowest one, and $v, w, s$ correspond to the next shaded unit triangles as we move counter-clockwise from the lowest one. We notice that the north side of the region has length $x+y+m \geqslant y+m \geqslant 2$ and the northeast side has length $t+a \geqslant 2$, so the four vertices $u, v, w, s$ are well-defined.

By removing the lozenges forced by the shaded unit triangles, we get back new $B$-type regions weighted by $wt_2$. Collecting the weights of those forced lozenges, we get

$$\mathrm{M}(G-\{u,v,w,s\}) = q^{\binom{z+m+1}{2}+(x+y+m-2)(z+t+m+a)}\,\mathrm{M}_2\left(B_{m,a}(x,y-1,z,t-1)\right), \tag{4.8}$$

$$\mathrm{M}(G-\{u,v\}) = q^{\binom{z+m+1}{2}}\,\mathrm{M}_2\left(B_{m,a}(x,y-1,z,t)\right), \tag{4.9}$$

$$\mathrm{M}(G-\{w,s\}) = q^{(x+y+m-2)(z+t+m+a)}\,\mathrm{M}_2\left(B_{m,a}(x,y,z,t-1)\right), \tag{4.10}$$

$$\mathrm{M}(G-\{u,s\}) = q^{\binom{z+m+1}{2}}\,\mathrm{M}_2\left(B_{m,a}(x,y-1,z+1,t-1)\right), \tag{4.11}$$

and

$$\mathrm{M}(G-\{v,w\}) = q^{(x+y+m-1)(z+t+m+a)}\,\mathrm{M}_2\left(B_{m,a}(x,y,z-1,t)\right) \tag{4.12}$$

(see Figures 4.3(b)–(f), respectively). Plugging the above identities into the equation (2.1) in Kuo Condensation Theorem 2.1, we obtain

$$\begin{aligned}\mathrm{M}_2\left(B_{m,a}(x,y,z,t)\right)\mathrm{M}_2\left(B_{m,a}(x,y-1,z,t-1)\right) = \\ \mathrm{M}_2\left(B_{m,a}(x,y-1,z,t)\right)\mathrm{M}_2\left(B_{m,a}(x,y,z,t-1)\right) \\ + q^{z+t+m+a}\,\mathrm{M}_2\left(B_{m,a}(x,y-1,z+1,t-1)\right)\mathrm{M}_2\left(B_{m,a}(x,y,z-1,t)\right).\end{aligned} \tag{4.13}$$

All regions in the above equation, except for the first one, have the sum of their $y$-, $z$- and $t$-parameters strictly less than $y+z+t$. Thus, by the induction hypothesis, those regions have their tiling generating functions given by (4.1). By substituting these formulas into the above equation and working on some simplifications, one readily gets $\mathrm{M}_2\left(B_{m,a}(x,y,z,t)\right)$ equal exactly to the expression on the right-hand side of (4.1). This finishes our proof. □

We need another $q$-enumeration of lozenge tilings of the magnet bar region as follows.

Assume that we now give all right and left lozenges in the magnet bar region $B_{m,a}(x,y,z,t)$ a weight 1. Next, we give a *vertical* lozenge a weight $q^l$, where $l$ is the distance between the *northeast* side of the lozenge and the *southwest* side of the region. We denote by $wt_3$ the new weight assignment, and $\mathrm{M}_3$ the corresponding tiling generating function.

**Proposition 4.2.** *For non-negative integers $m, a, x, y, z, t$*

$$\begin{aligned}\mathrm{M}_3\big(B_{m,a}(x,y,z,t)\big) &= q^{m\binom{a+1}{2}+t\binom{z+a+1}{2}+a(z+m)(x+a)+a\binom{z+m+1}{2}} \\ &\times \frac{\mathrm{H}_q(m+a+x+y+z+t)}{\mathrm{H}_q(m+a+x+y+t)\,\mathrm{H}_q(m+a+x+y+z)} \\ &\times \frac{\mathrm{H}_q(m+a+x+t)\,\mathrm{H}_q(m+a+x+y)\,\mathrm{H}_q(m+a+y+z)}{\mathrm{H}_q(m+a+z+t)\,\mathrm{H}_q(m+a+x)\,\mathrm{H}_q(m+a+y)} \\ &\times \frac{\mathrm{H}_q(x)\,\mathrm{H}_q(y)\,\mathrm{H}_q(z)\,\mathrm{H}_q(t)\,\mathrm{H}_q(m)\,\mathrm{H}_q(a)^2}{\mathrm{H}_q(a+x)\,\mathrm{H}_q(a+y)\,\mathrm{H}_q(z+t)\,\mathrm{H}_q(m+a)} \\ &\times \frac{\mathrm{H}_q(m+z+t)\,\mathrm{H}_q(m+a+x)\,\mathrm{H}_q(m+a+y)}{\mathrm{H}_q(m+y+z)\,\mathrm{H}_q(m+x+t)}.\end{aligned} \tag{4.14}$$

*Proof.* The equality (4.14) can be treated similarly to (4.1) in the Proposition 4.1 by induction on $y+z+t$. The base cases are the situations when $a=0$, $x=0$, $y=0$, $z=0$ or $t=0$.

Assume that our magnet bar region is weighted by $wt_3$[3].

If $a=0$, we get a weighted version of the hexagon $Hex(z, x+y+m, t)$ by removing forced lozenges from the magnet bar reigon as in Figure 4.2(e). Rotating the hexagon $60^\circ$ clockwise and reflecting it over a vertical line, we get the hexagon $Hex(z, t, x+y+m)$ weighted by $wt_2$, and (4.14) follows from Corolarry 3.2.

If $x=0$, after removing forced vertical lozenges (whose weight product is equal to $q^{(t+m)\binom{a+1}{2}}$) as in Figure 4.2(a), we get a weighted region $R$. Next, we rotate $R$ $60^\circ$ counter-clockwise and reflect it about a vertical line. This way, we get the weighted region $K_m(a,t,z,y)$ in which a right lozenge is weighted by $q^{a+l}$, where $l$ is the distance from the top of the lozenge to the bottom of the region. We divide the weight of each right lozenge in the latter region by $q^a$, and get back the weight assignment $wt_2$. Thus,

$$\mathrm{M}_3\left(B_{m,a}(0,y,z,t)\right) = q^{(t+m)\binom{a+1}{2}} q^{azt+a^2(z+m)}\, \mathrm{M}_2\left(K_m(a,t,z,y)\right), \tag{4.15}$$

where the factor $q^{azt+a^2(z+m)}$ comes from the weight division. Then (4.14) follows from Lemma 3.3.

If $y=0$, by removing forced lozenges (whose weight product is $q^{(x+a)(z+m)a+a\binom{z+m+1}{2}}$) and rotating the resulting region $60^\circ$ clockwise, we get a weighted version of region $K_m(a,z,t,x)$ in which a right lozenge is weighted by $q^{m+a+x+z+1-l}$, where $l$ is the distance from the left side of the lozenge and the southeast side of the region (see Figure 4.2(b)). By dividing the weight of each right lozenge by $q^{m+a+x+z+1}$, we get back the weight assignment $wt_1$, *where* $q$ *is replaced by* $q^{-1}$. Thus, (4.14) follows from Proposition 3.1, Lemma 3.3 and the simple fact $[n]_{q^{-1}} = [n]_q/q^{(n-1)}$.

If $z=0$, we apply Region-splitting Lemma 2.2 (and remove forced lozenges weighted by 1) to split our region into two hexagons as in Figure 4.2(c). Next, we rotate the right hexagon $60^\circ$ counter-clockwise and reflect it about a vertical line to get the hexagon $Hex(a, t+m, x)$ weighted by $wt_2$. For the left hexagon, we also rotate it $60^\circ$ counter-clockwise, reflect it about a vertical line and divide the weight of each right lozenge of the resulting region by $q^{x+a}$ to get the hexagon $Hex(m,a,y)$ weighted by $wt_2$. Then we get (4.14) from Corollary 3.2. The case $t=0$, can be treated similarly to the case $z=0$, based on Figure 4.2(d).

The induction step is completely analogous to that of the proof of Proposition 4.1. We also apply Kuo's Theorem 2.1, based on Figure 4.3. After removing lozenges forced by the shaded unit triangles, we get back new $B$-type regions weighted by $wt_3$. Figure 4.3 tells us that the product of $\mathrm{M}_3$-generating functions of two regions on the top is equal to the product of the $\mathrm{M}_3$-generating functions of two regions in the middle plus the product of $\mathrm{M}_3$-generating functions of two regions on the bottom. To precise, we get the following recurrence

$$\begin{aligned}\mathrm{M}_3\left(B_{m,a}(x,y,z,t)\right)\mathrm{M}_3\left(B_{m,a}(x,y-1,z,t-1)\right) = \\ \mathrm{M}_3\left(B_{m,a}(x,y-1,z,t)\right)\mathrm{M}_3\left(B_{m,a}(x,y,z,t-1)\right) \\ + q^{m+a+x+y+z}\,\mathrm{M}_3\left(B_{m,a}(x,y-1,z+1,t-1)\right)\mathrm{M}_3\left(B_{m,a}(x,y,z-1,t)\right),\end{aligned} \tag{4.16}$$

and the proposition follows from the induction hypothesis. □

[3]We still follow the process in Figures 4.2 and 4.3, however, the reader should be aware that the weight assignment here is *different* from that in the proof of Proposition 4.1.

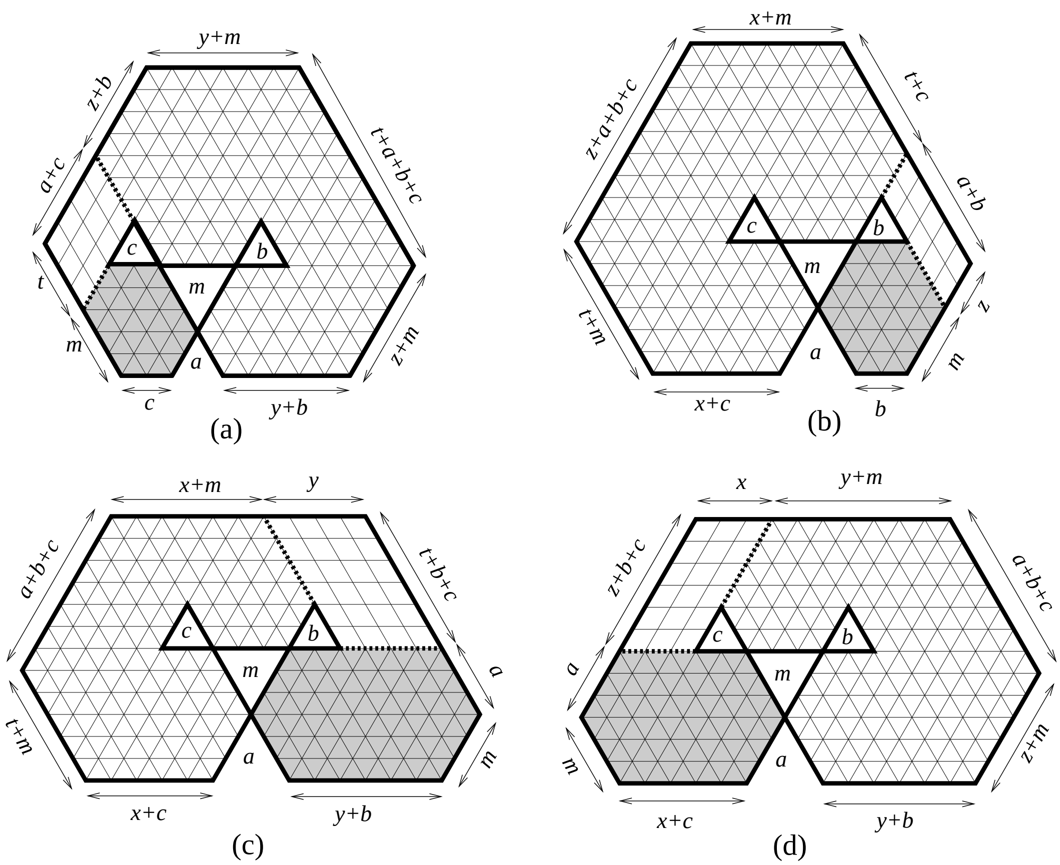



Figure 5.1: The four base cases: (a) $x = 0$, (b) $y = 0$, (c) $z = 0$ (b), and $t = 0$.

# 5 Proof of Theorem 1.2

By Proposition 3.1, we only need to show that

$$
\begin{aligned}
&\mathrm{M}_2\left(Q\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}\right) = \\
&\quad q^{\mathbf{g}\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}} \frac{\mathrm{H}_q(m+a+b+c+x+y+z+t)}{\mathrm{H}_q(m+a+b+c+x+y+t)\,\mathrm{H}_q(m+a+b+c+x+y+z)}\\
&\quad \times \frac{\mathrm{H}_q(m+a+b+c+x+t)\,\mathrm{H}_q(m+a+b+c+x+y)\,\mathrm{H}_q(m+a+b+c+y+z)}{\mathrm{H}_q(m+a+b+c+z+t)\,\mathrm{H}_q(m+a+b+c+x)\,\mathrm{H}_q(m+a+b+c+y)}\\
&\quad \times \frac{\mathrm{H}_q(x)\,\mathrm{H}_q(y)\,\mathrm{H}_q(z)\,\mathrm{H}_q(t)}{\mathrm{H}_q(x+t)\,\mathrm{H}_q(y+z)} \frac{\mathrm{H}_q(m)^3\,\mathrm{H}_q(a)^2\,\mathrm{H}_q(b)\,\mathrm{H}_q(c)\,\mathrm{H}_q(m+a+b+c)}{\mathrm{H}_q(m+a)^2\,\mathrm{H}_q(m+b)\,\mathrm{H}_q(m+c)}\\
&\quad \times \frac{\mathrm{H}_q(m+b+c+z+t)\,\mathrm{H}_q(m+a+c+x)\,\mathrm{H}_q(m+a+b+y)}{\mathrm{H}_q(m+b+y+z)\,\mathrm{H}_q(m+c+x+t)}\\
&\quad \times \frac{\mathrm{H}_q(c+x+t)\,\mathrm{H}_q(b+y+z)}{\mathrm{H}_q(a+c+x)\,\mathrm{H}_q(a+b+y)\,\mathrm{H}_q(b+c+z+t)}. \qquad (5.1)
\end{aligned}
$$

We prove (5.1) by induction on $y+z+t$. The base cases here are the situations when one of the four parameters $x, y, z, t$ equals to 0.

We assume that our region $Q := Q\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}$ is weighted by $wt_2$

If $x = 0$, by applying Region-splitting Lemma 2.2, we split the into two parts as in Figure 5.1(a): the shaded hexagon $Hex(a, c, m)$ (weighted by $wt_2$) and $Q - Hex(a, c, m)$. We remove forced vertical lozenges (which have the weight 1) from the latter region, then rotate the resulting region 60° counter-clockwise, and reflect it about a vertical line. This way we get the magnet bar region $B_{b,m}(a, t + c, z, y)$ weighted by $wt_3$. Thus, we have

$$\begin{aligned}\mathrm{M}_2\left(Q\begin{pmatrix} 0 & y & z & t\\ m & a & b & c\end{pmatrix}\right) &= \mathrm{M}_2\left(Hex(a,c,m)\right)\mathrm{M}_3\left(Q\begin{pmatrix} 0 & y & z & t\\ m & a & b & c\end{pmatrix} - Hex(a,c,m)\right)\\ &= \mathrm{M}_2\left(Hex(a,c,m)\right)\mathrm{M}_3\left(B_{b,m}(a,t+c,z,y)\right),\end{aligned} \tag{5.2}$$

and (5.1) follows from Corollary 3.2 and Proposition 4.2. The case $t = 0$ can be treated similarly to the case $x = 0$, based on Figure 5.1(d). The only difference is that our forced right lozenges have weight product equal to $q^{x(a+m)(z+b+c)+x\binom{z+b+c+1}{2}}$. Thus, we get

$$\begin{aligned}&\mathrm{M}_2\left(Q\begin{pmatrix} x & y & z & 0\\ m & a & b & c\end{pmatrix}\right)\\ &= \mathrm{M}_2\left(Hex(a,x+c,m)\right)\mathrm{M}_2\left(Q\begin{pmatrix} x & y & z & 0\\ m & a & b & c\end{pmatrix} - Hex(a,x+c,m)\right)\\ &= \mathrm{M}_2\left(Hex(a,x+c,m)\right)q^{x(a+m)(z+b+c)+x\binom{z+b+c+1}{2}}\mathrm{M}_3\left(B_{b,m}(a,c,z,y)\right).\end{aligned} \tag{5.3}$$

Again, (5.1) follows from the Corollary 3.2 and Proposition 4.2.

If $y = 0$, again, by Region-splitting Lemma 2.2, we get

$$\mathrm{M}_2\left(Q\begin{pmatrix} x & 0 & z & t\\ m & a & b & c\end{pmatrix}\right) = \mathrm{M}_2\left(Hex(m,b,a)\right)\mathrm{M}_2\left(Q\begin{pmatrix} x & 0 & z & t\\ m & a & b & c\end{pmatrix} - Hex(m,b,a)\right) \tag{5.4}$$

(see Figure 5.1(b)). We also remove forced lozenges (having weight 1) from the second region on the right-hand side to get a region $R'$. Next, we rotate $R'$ 60° clockwise and reflect it about a vertical line to get the magnet bar $B_{c,m}(b + z, a, x, t)$ weighted by $wt_1$. Thus, we have

$$\mathrm{M}_2\left(Q\begin{pmatrix} x & 0 & z & t\\ m & a & b & c\end{pmatrix}\right) = \mathrm{M}_2\left(Hex(m,b,a)\right)\mathrm{M}_1\left(B_{c,m}(b+z,a,x,t)\right), \tag{5.5}$$

and (5.1) follows from Corollary 3.2 and Propositions 3.1 and 4.1. The case $z = 0$ can be obtained in the same way, based on Figure 5.1(c).

For induction step, we assume that $x, y, z, t$ are positive and that (5.1) holds for any $Q$-type regions in which the sum of the $y$-, $z$- and $t$-parameters strictly less than $y + z + t$.

We now apply Kuo condensation to the dual graph $G$ of the region $Q\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}$ weighted by $wt_2$. The four vertices $u, v, w, s$ correspond to the four shaded unit triangles in Figure 5.2(b). By collecting the weights of the forced lozenges shown in Figures 5.2(b)–(f), we get respectively

$$\mathrm{M}(G-\{u,v,w,s\}) = q^{\binom{z+m+1}{2}+(x+y+m-2)(z+t+m+a+b+c)}\mathrm{M}_2\left(Q\begin{pmatrix} x & y-1 & z & t-1\\ m & a & b & c\end{pmatrix}\right), \tag{5.6}$$

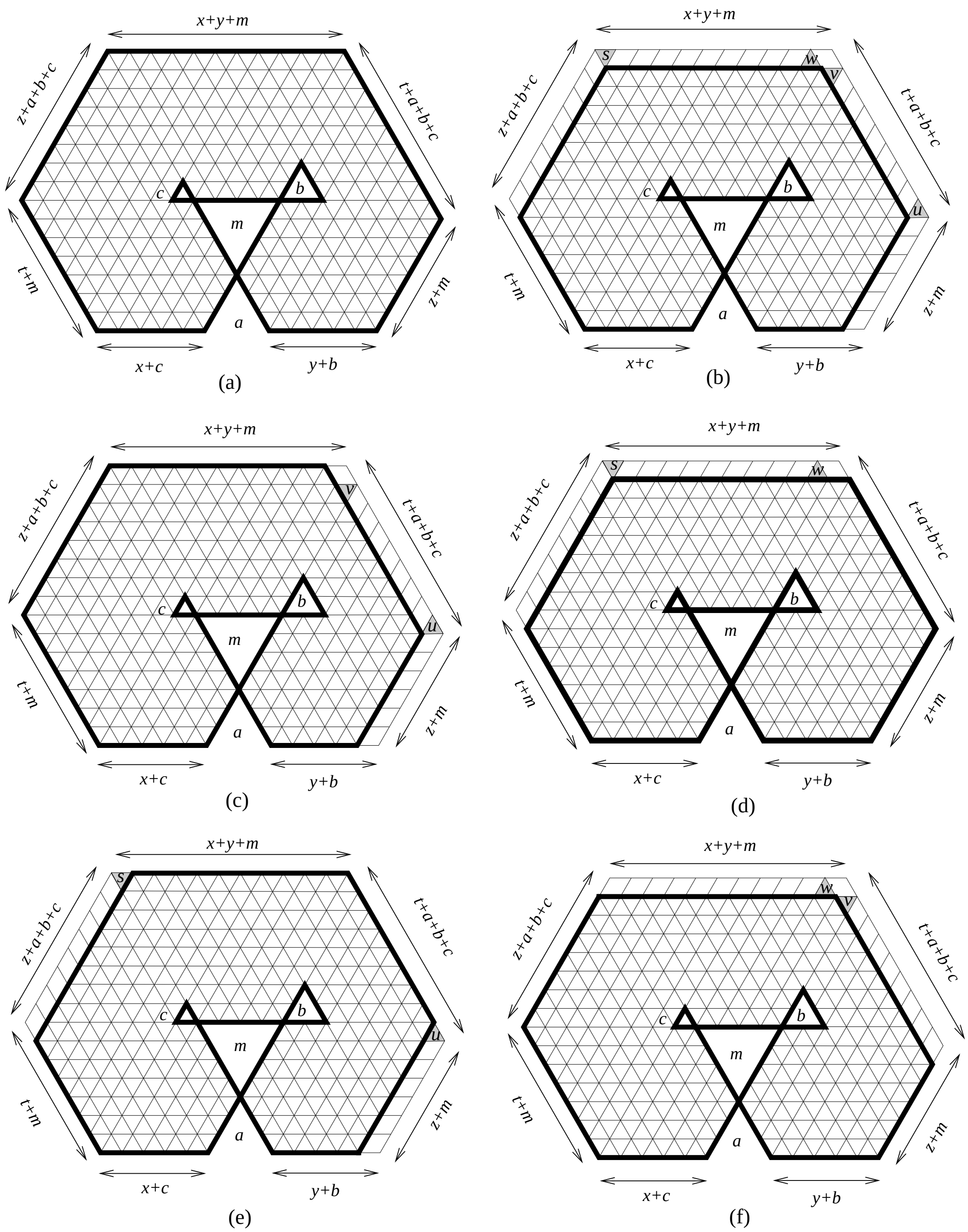



Figure 5.2: Obtaining the recurrence on the numbers of tilings by using Kuo's condensation.

$$\mathrm{M}(G-\{u,v\}) = q^{\binom{z+m+1}{2}}\,\mathrm{M}_2\left(Q\begin{pmatrix} x & y-1 & z & t\\ m & a & b & c\end{pmatrix}\right), \tag{5.7}$$

$$\mathrm{M}(G-\{w,s\}) = q^{(x+y+m-2)(z+t+m+a+b+c)}\,\mathrm{M}_2\left(Q\begin{pmatrix} x & y & z & t-1\\ m & a & b & c\end{pmatrix}\right), \tag{5.8}$$

$$\mathrm{M}(G-\{u,s\}) = q^{\binom{z+m+1}{2}}\,\mathrm{M}_2\left(Q\begin{pmatrix} x & y-1 & z+1 & t-1\\ m & a & b & c\end{pmatrix}\right), \tag{5.9}$$

and

$$\mathrm{M}(G-\{v,w\}) = q^{(x+y+m-1)(z+t+m+a+b+c)}\,\mathrm{M}_2\left(Q\begin{pmatrix} x & y & z-1 & t\\ m & a & b & c\end{pmatrix}\right). \tag{5.10}$$

Substituting (5.6)–(5.10) into equation (2.1) in Kuo's Theorem 2.1, we get

$$\begin{aligned}
&\mathrm{M}_2\left(Q\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}\right)\mathrm{M}_2\left(Q\begin{pmatrix} x & y-1 & z & t-1\\ m & a & b & c\end{pmatrix}\right)\\
&\quad= \mathrm{M}_2\left(Q\begin{pmatrix} x & y-1 & z & t\\ m & a & b & c\end{pmatrix}\right)\mathrm{M}_2\left(Q\begin{pmatrix} x & y & z & t-1\\ m & a & b & c\end{pmatrix}\right)\\
&\quad+ q^{z+t+m+a+b+c}\,\mathrm{M}_2\left(Q\begin{pmatrix} x & y-1 & z+1 & t-1\\ m & a & b & c\end{pmatrix}\right)\mathrm{M}_2\left(Q\begin{pmatrix} x & y & z-1 & t\\ m & a & b & c\end{pmatrix}\right).
\end{aligned} \tag{5.11}$$

Finally, if we denote by $\Psi\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}$ the expression on the right-hand side of (5.1), we only need to show that $\Psi$ satisfies also the recurrence (5.11). Equivalently, we need to verify that

$$\begin{aligned}
&\frac{\Psi\begin{pmatrix} x & y-1 & z & t\\ m & a & b & c\end{pmatrix}}{\Psi\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}}\,\frac{\Psi\begin{pmatrix} x & y & z & t-1\\ m & a & b & c\end{pmatrix}}{\Psi\begin{pmatrix} x & y-1 & z & t-1\\ m & a & b & c\end{pmatrix}}+\\
&\qquad q^{z+t+m+a+b+c}\frac{\Psi\begin{pmatrix} x & y & z-1 & t\\ m & a & b & c\end{pmatrix}}{\Psi\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}}\,\frac{\Psi\begin{pmatrix} x & y-1 & z+1 & t-1\\ m & a & b & c\end{pmatrix}}{\Psi\begin{pmatrix} x & y-1 & z & t-1\\ m & a & b & c\end{pmatrix}}=1.
\end{aligned} \tag{5.12}$$

Let $\Phi\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix} := q^{-\mathbf{g}\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}}\Psi\begin{pmatrix} x & y-1 & z & t\\ m & a & b & c\end{pmatrix}$. By the definition of the function $\mathbf{g}$, we get

$$\mathbf{g}\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}+\mathbf{g}\begin{pmatrix} x & y & z & t-1\\ m & a & b & c\end{pmatrix}=\mathbf{g}\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}+\mathbf{g}\begin{pmatrix} x & y-1 & z & t-1\\ m & a & b & c\end{pmatrix} \tag{5.13}$$

and

$$\begin{aligned}
&\mathbf{g}\begin{pmatrix} x & y & z-1 & t\\ m & a & b & c\end{pmatrix}+\mathbf{g}\begin{pmatrix} x & y & z-1 & t\\ m & a & b & c\end{pmatrix}\\
&\qquad=(m+x+y-z-1)+\mathbf{g}\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}+\mathbf{g}\begin{pmatrix} x & y-1 & z & t-1\\ m & a & b & c\end{pmatrix}.
\end{aligned} \tag{5.14}$$

Therefore, (5.12) is equivalent to

$$\frac{\Phi\begin{pmatrix} x & y-1 & z & t\\ m & a & b & c\end{pmatrix}}{\Phi\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}}\frac{\Phi\begin{pmatrix} x & y & z & t-1\\ m & a & b & c\end{pmatrix}}{\Phi\begin{pmatrix} x & y-1 & z & t-1\\ m & a & b & c\end{pmatrix}}+ q^{m+a+b+c+x+y+t-1}\frac{\Phi\begin{pmatrix} x & y & z-1 & t\\ m & a & b & c\end{pmatrix}}{\Phi\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}}\frac{\Phi\begin{pmatrix} x & y-1 & z+1 & t-1\\ m & a & b & c\end{pmatrix}}{\Phi\begin{pmatrix} x & y-1 & z & t-1\\ m & a & b & c\end{pmatrix}}=1. \tag{5.15}$$

Let us simplify the first term on the left-hand side of (5.15). We notice that the two $\Phi$-functions in the numerator and the denominator of the first fraction in the first term are different only at their $y$-parameters. Canceling out all terms, which have no $y$-parameter, and using the trivial fact $\mathrm{H}_q(n+1)/\mathrm{H}_q(n)=[n]_q!$, we get

$$\frac{\Phi\begin{pmatrix} x & y-1 & z & t\\ m & a & b & c\end{pmatrix}}{\Phi\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}}=\frac{[y+z-1]_q![a+b+y-1]_q![m+b+y+z-1]_q!}{[y-1]_q![b+y+z-1]_q![m+a+b+y-1]_q!}$$
$$\times\frac{[m+a+b+c+y-1]_q![m+a+b+c+x+y+t-1]_q![m+a+b+c+x+y+z-1]_q!}{[m+a+b+c+x+y-1]_q![m+a+b+c+y+z-1]_q![m+a+b+c+x+y+z+t-1]_q!}. \tag{5.16}$$

Doing similarly for the second fraction of the first term, we obtain

$$\frac{\Phi\begin{pmatrix} x & y & z & t-1\\ m & a & b & c\end{pmatrix}}{\Phi\begin{pmatrix} x & y-1 & z & t-1\\ m & a & b & c\end{pmatrix}}=\frac{[y-1]_q![b+y+z-1]_q![m+a+b+y-1]_q!}{[y+z-1]_q![a+b+y-1]_q![m+b+y+z-1]_q!}$$
$$\times\frac{[m+a+b+c+x+y-1]_q![m+a+b+c+y+z-1]_q![m+a+b+c+x+y+z+t-2]_q!}{[m+a+b+c+y-1]_q![m+a+b+c+x+y+t-2]_q![m+a+b+c+x+y+z-1]_q!}. \tag{5.17}$$

This implies that the first term on the left-hand side of (5.15) can be simplified as

$$\frac{\Phi\begin{pmatrix} x & y-1 & z & t\\ m & a & b & c\end{pmatrix}}{\Phi\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}}\frac{\Phi\begin{pmatrix} x & y & z & t-1\\ m & a & b & c\end{pmatrix}}{\Phi\begin{pmatrix} x & y-1 & z & t-1\\ m & a & b & c\end{pmatrix}}=\frac{[m+a+b+c+x+y+t-1]_q}{[m+a+b+c+x+y+z+t-1]_q}. \tag{5.18}$$

We simplify the second term on the left-hand side of (5.15) in the same way (the numerator and the denominator in each fraction are now different at their $z$-parameters). We get

$$\frac{\Phi\begin{pmatrix} x & y & z-1 & t\\ m & a & b & c\end{pmatrix}}{\Phi\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}}\frac{\Phi\begin{pmatrix} x & y-1 & z+1 & t-1\\ m & a & b & c\end{pmatrix}}{\Phi\begin{pmatrix} x & y-1 & z & t-1\\ m & a & b & c\end{pmatrix}}=\frac{[z]_q}{[m+a+b+c+x+y+z+t-1]_q}. \tag{5.19}$$

By (5.18) and (5.19), the equality (5.15) becomes the following identity

$$\frac{[m+a+b+c+x+y+t-1]_q+q^{m+a+b+c+x+y+t-1}[z]_q}{[m+a+b+c+x+y+z+t-1]_q}=1, \tag{5.20}$$

which follows directly from the definition of the $q$-integers. This completes our proof.

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